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On couple-stress fluid permeated with suspended particles heated and soluted from below in porous medium

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Abstract : A layer of couple-stress fluid, permeated with suspended particles, heated and soluted from below in porous medium is considered. The couple-stress and stable solute gradient postpone the onset of convection whereas the medium permeability and suspended particles hasten the onset of convection. The principle of exchange of stabilities is valid for the couple-stress fluid permeated with suspended particles heated from below in porous medium. The oscillatory modes are introduced due to the presence of stable solute gradient.

Keywords : Thermosolutal convection, couple-stress fluid, suspended particles, porous medium.

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1. Introduction

The theory of Bénard convection in a viscous, Newtonian fluid layer heated from below has been given by Chandrasekhar [1]. Chandra [2] observed that in an air layer, convection occurred at much lower gradients than predicted if the layer depth was less than 7 mm and called this motion, 'columnar instability'. However, for layers er than 10 mm, a Bénard-type cellular convection was ved. Thus there is a contradiction between the theory l the experiment. Scanlon and Segel [3] have considered effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure fluid was supplemented by that of the particles. laniswamy and Purushotham [4] have considered the ability of shear flow of stratified fluid with fine dust and ave found the effect of fine dust to increase the region instability.

The study of a layer of fluid heated from below in ous medium is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines, one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals. The basic equations of a layer of fluid heated from below in porous medium have been derived by Joseph [5].

The importance of non-Newtonian fluids in modern technology and industries is ever increasing and the investigations on such fluids are desirable. Stokes [6] has formulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. The shoulder, knee, hip and ankle joints are the loaded-bearing synovial joints of the human body and these joints have a low-friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. Walicki and Walicka [7] have modelled synovial fluid as a couple-stress fluid in human joints.

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Environmental pollution is the main cause of dust to enter into the human body. The metal dust which filters into the blood stream of those working near the furnace causes extensive damage to the chromozomes and genetic mutation so observed are likely to breed cancer or malformations in the coming progeny. Therefore, it is very essential to study the blood flow with dust particles. Considering blood as couple-stress fluid and dust particles as micro-organisms, Rathod and Thippeswamy [8] have studied the gravity flow of pulsatile blood through closed rectangular inclined channel with micro-organisms.

The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been investigated by Veronis [9]. The physics in the steller case is quite similar to Veronis [9] thermohaline configurations, in that helium acts like salt in raising the density and in diffusing more slowly than heat. This problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its application to oceanography and astrophysics. The heat and solute being two diffusing components, thermosolutal convection is the general term dealing with such phenomena. The fluid has been assumed to be Newtonian by Veronis.

The present paper attempts to study the effect of suspended (or dust) particles on the couple-stress fluid heated and soluted from below in porous medium, keeping in mind the importance and applications of non-Newtonian fluids, suspended particles, porous medium and convection in fluid layer heated and soluted from below.

2. Formulation of the problem and perturbation equations

Consider an infinite horizontal couple-stress fluid-particle layer of thickness d bounded by the planes $z = 0$ and $z = d$. This layer is heated and soluted from below so that the temperatures, densities and solute concentrations at the bottom surface $z = 0$ are T_0, ρ_0, C_0 and at the upper surface $z = d$ are T_d, ρ_d, C_d respectively and that a uniform temperature gradient $\beta(=|dT/dz|)$ and solute gradient β' $(=|dT/dz|)$ are maintained. Let ρ, p, T and $q(u,v,w)$ denote respectively the density, pressure, temperature and filter velocity. $q_d(\bar{x},t)$ and N $(\bar{x},t)$ denote the velocity and number density of the suspended particles respectively. Then the momentum balance and mass balance equations of the couple-stress fluid through porous medium (Stokes [6], Joseph [5], Scanlon and Segel [3]) are

$$\frac{\partial q}{\partial t}+\frac{1}{\in}(q.\nabla)q = -\frac{1}{\rho_0}\nabla p - g\left(1+\frac{\delta\rho}{\rho_0}\right)\lambda$$

$$-\frac{1}{k_1}\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)q+\frac{KN}{\rho_0\in}\left(q_d-q\right) \tag{1}$$

$$\nabla.q = 0. \tag{2}$$

The equation of state for the fluid is

$$\rho=\rho_0[1-\alpha(T-T_0)+\alpha'(C-C_0)]\ , \tag{3}$$

where suffix zero refers to the values at the reference level $z = 0$ and α, α' are the coefficients of thermal expansion and analogous solvent expansion respectively. Here, k_1 is the medium permeability, $\in$ is the medium porosity, $g=(0,0,-g)$ is acceleration due to gravity, $\bar{x}=(x,y,z)$, $\lambda = (0,0,1)$ and $K=6\pi\mu\eta'$, (η' being particle radius), is the Stokes' drag coefficient. Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term, in the equations of motion (1), proportional to the velocity difference between particles and fluid.

The force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid. Inter-particle reactions are ignored for we assume that the distances between particles are quite large compared with their diameters. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles are

$$mN\frac{\partial q_d}{\partial t}+\frac{1}{\in}(q_d.\nabla)q_d = KN(q-q_d), \tag{4}$$

$$\in\frac{\partial N}{\partial t}+\nabla.(Nq_d)=0. \tag{5}$$

Let c_v, c_{pt}, T and q' denote, respectively, the specific h of the fluid at constant volume, the specific heat of the particles, the temperature and the 'effective thermal conductivity' of the pure fluid. Assuming that the particles and the fluid are in thermal equilibrium, the equatior heat conduction gives

$$[\rho_0 c_v \in +\rho_s c_s(1-\in)]\frac{\partial T}{\partial t} + \rho_0 c_v (q.\nabla)T +$$

$$mNc_{pt}\left[\in \frac{\partial}{\partial t} + q_d.\nabla\right]T = q'\nabla^2 T, \tag{6}$$

where ρ_s, c_s are the density and the specific heat of the solid (porous matrix) material respectively. The kinematic viscosity ν, couple-stress viscosity μ', coefficient of thermal expansion α and the thermal conductivity q' are all assumed to be constants.

If C denotes the solute concentration, then the equation of solute conduction gives

$$[\rho_0 c_v' \in +\rho_s c_s(1-\in)]\frac{\partial C}{\partial t} + \rho_0 c_v' (q.\nabla)C +$$

$$mNc'_{pt}\left[\in \frac{\partial}{\partial t} + q_d.\nabla\right]C = q''\nabla^2 C, \tag{7}$$

where c_v', c_{pt}', q'' are the analogous solute quantities. The basic motionless solution is

$$q = (0,0,0),\ q_d = (0,0,0),\ T = T_0 - \beta z,\ C = C_0 - \beta' z,$$

$$\rho = \rho_0(1 + \alpha\beta z - \alpha'\beta' z),\ N = N_0,\ \text{a constant.} \tag{8}$$

Assume small perturbations around the basic solution and let $\delta\rho$, N, δp, θ, γ, $q(u,v,w)$ and $q_d(l,r,s)$ denote respectively the perturbations in density ρ, suspended particles number density N_0, pressure p, temperature T, solute concentration C, couple-stress fluid velocity (0,0,0) and particle velocity (0,0,0). The change in density $\delta\rho$, caused mainly by the perturbation θ and γ in temperature and solute concentration, is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \tag{9}$$

Then, the linearized perturbation equations of couple-stress fluid and suspended particles are

$$\frac{1}{\in}\frac{\partial q}{\partial t} = -\frac{1}{\rho_0}\nabla\delta p + g(\alpha\theta - \alpha'\gamma)\lambda - \frac{1}{k_1}$$

$$\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)q + \frac{KN_0}{\rho_0}(q_d - q), \tag{10}$$

$$\nabla.q = 0, \tag{11}$$

$$mN_0\frac{\partial q_d}{\partial t} = KN_0(q - q_d), \tag{12}$$

$$(E + h\in)\frac{\partial\theta}{\partial t} = \beta(w + hs) + \kappa\nabla^2\theta, \tag{13}$$

$$(E' + h'\in)\frac{\partial\gamma}{\partial t} = \beta'(w + h's) + \kappa'\nabla^2\gamma, \tag{14}$$

where

$$E = \in + (1-\in)\frac{\rho_s c_s}{\rho_0 c_v},\quad \kappa = \frac{q}{\rho_0 c_v}\quad \text{and}\quad h = \frac{mN_0 c_{pt}}{\rho_0 c_v}$$

$$E' = \in + (1-\in)\frac{\rho_s c_s}{\rho_0 c_v'},\ \kappa' = \frac{q'}{\rho_0 c_v'}.\ \text{and}\ h' = \frac{mN_0 c_{pt}'}{\rho_0 c_v'}$$

Eliminating q_d in eq. (10) with the help of eq. (12), writing the scalar components of eq. (10) and eliminating u, v, δp between them by using eq. (11), we obtain

$$\frac{1}{\in}\frac{\partial}{\partial t}\nabla^2 w + \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2 w - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g($$

$$g(\alpha\theta - \alpha'\gamma) - \frac{KN_0}{\rho_0\in}\left[\frac{1}{\frac{m}{K}\frac{\partial}{\partial t} + 1} - 1\right]\nabla^2 w = 0. \tag{15}$$

Eliminating s with the help of eq. (12), eqs. (13) and (14) yield

$$\left(\frac{m}{K}\frac{\partial}{\partial t} + 1\right)\left[\overline{E + h\in}\frac{\partial}{\partial t} - \kappa\nabla^2\right]\theta = \beta\left(\frac{m}{K}\frac{\partial}{\partial t} + 1 + h\right)w \tag{16}$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t} + 1\right)\left[\overline{E' + h'\in}\frac{\partial}{\partial t} - \kappa\nabla^2\right]\gamma$$

$$= \beta'\left(\frac{m}{K}\frac{\partial}{\partial t}+1+h'\right)w. \tag{17}$$

3. The dispersion relation

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w,\theta,\gamma] = [W(z),\Theta(z),\Gamma(z)]e^{ik_x x + ik_y y + nt} \tag{18}$$

where k_x, k_y are wave numbers along the x- and y-directions respectively, $k\left(=\sqrt{k_x^2+k_y^2}\right)$ is the resultant wave number and n is, in general, a complex constant.

Expressing the coordinates x, y, z in the new unit of length d, time t in the new unit of length $\frac{d^2}{\kappa}$ and letting

$$a = kd,\ \sigma = \frac{nd^2}{\nu},\ p_1 = \frac{\nu}{\kappa},\ P_l = \frac{k_1}{d^2},\ F = \frac{\mu'/\rho_0 d^2}{\nu},$$

$$p_1' = \frac{\nu}{\kappa'},\quad n' = n\left[1+\frac{mN_0K}{\rho_0(mn+K)}\right]\quad \sigma' = \frac{n'd^2}{\nu},\ H = h+1,$$

$$H' = h'+1,\ \tau = \frac{m\kappa}{Kd^2} \text{ and } D = \frac{d}{dz}.$$

Eqs. (15), (16) and (17) using (18) yield

$$\left[\frac{\sigma'}{\epsilon}+\frac{1}{P_l}\left\{1-F\left(D^2-a^2\right)\right\}\right]\left(D^2-a^2\right)W$$

$$= -\frac{ga^2d^2}{\nu}\left(\alpha\Theta-\alpha'\Gamma\right), \tag{19}$$

$$\left(\frac{\tau\nu\sigma}{d^2}+1\right)\left(D^2-a^2-\overline{E+h\epsilon}\,p_1\sigma\right)\Theta$$

$$= -\frac{\beta d^2}{\kappa}\left(H+\frac{\tau\nu\sigma}{d^2}\right)W, \tag{20}$$

$$\left(\frac{\tau\nu\sigma}{d^2}+1\right)\left(D^2-a^2-\overline{E'+h'\epsilon}\,p_1'\sigma\right)\Gamma$$

$$\beta' d^2 \left(H' + \frac{\tau\nu\sigma}{d^2} \right) W \tag{21}$$

Eliminating Θ and Γ between eqs. (19), (20) and (21), we obtain

$$\left(1+\frac{\tau\nu\sigma}{d^2}\right)\left[\frac{\sigma'}{\epsilon}+\frac{1}{P_l}\left\{1-F\left(D^2-a^2\right)\right\}\right]\left(D^2-a^2\right)$$

$$\left(D^2-a^2-\overline{E+h\epsilon}\,p_1\sigma\right)\left(D^2-a^2-\overline{E'+h'\epsilon}\,p_1'\sigma\right)W$$

$$= Ra^2\left(H+\frac{\nu\tau\sigma}{d^2}\right)\left(D^2-a^2-\overline{E'+h'\epsilon}\,p_1'\,\sigma\right)W - R'a^2$$

$$\left(H'+\frac{\tau\nu\sigma}{d^2}\right)\left(D^2-a^2-\overline{E+h\epsilon}\,p_1\sigma\right)W, \tag{22}$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the thermal Rayleigh number.

and $R' = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$ is the analogous solute Rayleigh number.

The perturbations in temperature and solute concentration are zero at the boundaries, since both the boundaries are maintained at constant temperatures and concentration. Consider the case of two free boundaries which are perfect conductors of heat, though little artificial, but it enables us to find analytical solutions. The appropriate boundary conditions with respect to which eq.(22) must be solved, are

$$W = D^2W = 0,\ \Theta = \Gamma = 0 \text{ at } z = 0 \text{ and } 1. \tag{23}$$

Using the boundary conditions (23), it can be shown that all the even order derivatives of W must vanish for $z = 0$ and $z = 1$ and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{24}$$

where W_0 is a constant. Substituting the proper solution (24) in eq. (22), we obtain

$$R_1 = \frac{\left(1+\frac{i\nu\tau\sigma_1\pi^2}{d^2}\right)\left[\frac{i\sigma_1'}{\epsilon}+\frac{1}{P}\{1+\pi^2F(1+x)\}\right.(1+x)\left(1+x+\overline{E+h\epsilon}\, ip_1\sigma_1\right)\left(1+x+\overline{E'+h'\epsilon}\, ip_1'\sigma_1\right)+R_1'\,x\left(H'+\frac{i\nu\tau\sigma_1\pi^2}{d^2}\right)\left(1+x+\overline{E+h\epsilon}\, ip_1\sigma_1\right)}{x\left(H+\frac{i\nu\tau\sigma_1\pi^2}{d^2}\right)\left(1+x+\overline{E'+h'\epsilon}\, ip_1'\sigma_1\right)} \tag{25}$$

where $R_1 = \frac{R}{\pi^4}, R_1' = \frac{R'}{\pi^4}\ x = \frac{a^2}{\pi^2},\ i\sigma_1 = \frac{\sigma}{\pi^2}, P = \pi^2 P_l$

and $\sigma_1' = \frac{\sigma'}{\pi^2}$.

4. The stationary convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, eq. (25) yields

$$R_1 = \frac{\frac{1}{P}\left[1+\pi^2F(1+x)\right](1+x)^2 + R_1'\, xH'}{xH} \tag{26}$$

Eq. (26) yields

$$\frac{dR_1}{dF} = \frac{(1+x)^3\pi^2}{xHP} \tag{27}$$

$$\frac{dR_1}{dP} = -\frac{(1+x)^2(1+\pi^2F(1+x))}{xHP^2}. \tag{28}$$

$$\frac{dR_1}{dH} = -\frac{\frac{1}{P}\left[1+\pi^2F(1+x)\right](1+x)^2 + R_1'\, xH'}{xH^2} \tag{29}$$

$$\frac{dR_1}{dR_1'} = \frac{H'}{H}. \tag{30}$$

It is clear from eqs. (27)–(30) that for stationary convection, the couple-stress and stable solute gradient postpone the onset of convection whereas the medium permeability and suspended particles hasten the onset of convection, on the couple-stress fluid permeated with suspended particles, heated and soluted from below in porous medium. The stable solute gradient has a stabilizing effect on the system.

5. Oscillatory modes

We shall show that for the problem under consideration, the principle of exchange of stabilities is not valid and oscillatory modes come into play.

Multiplying eq. (19) by W^*, the complex conjugate of W, integrating over the range of z and using (20), (21) together with boundary conditions (23), we get

$$\frac{F}{P_l}I_1 + \left(\frac{\sigma'}{\epsilon}+\frac{1}{P_l}\right)I_2 = \frac{g\alpha\kappa a^2}{\nu\beta}\left(\frac{d^2+\nu\tau\sigma^*}{Hd^2+\nu\tau\sigma^*}\right)\left(I_3+\overline{E+h\epsilon}\,p_1\sigma^* I_4\right) - \frac{g\alpha'\kappa' a^2}{\nu\beta'}\left(\frac{d^2+\nu\tau\sigma^*}{H'd^2+\nu\tau\sigma^*}\right)\left(I_5+\overline{E'+h'\epsilon}\,p_1'\sigma^* I_6\right) \tag{31}$$

where

$$\begin{aligned}
I_1 &= \int_0^1\left(\left|D^2W\right|^2 + 2a^2|DW|^2 + a^4|W|^2\right)dz,\\
I_2 &= \int_0^1\left(|DW|^2 + a^2|W|^2\right)dz,\\
I_3 &= \int_0^1\left(|D\Theta|^2 + a^2|\Theta|^2\right)dz,\\
I_4 &= \int_0^1\left(|\Theta|^2\right)dz,\\
I_5 &= \int_0^1\left(|D\Gamma|^2 + a^2|\Gamma|^2\right)dz\\
I_6 &= \int_0^1\left(|\Gamma|^2\right)dz.
\end{aligned} \tag{32}$$

Putting $\sigma = i\sigma_i$ and equating imaginary parts of eq. (31), we obtain

$$i\sigma_i\left[\left(1+\frac{f}{1+p_1^2\tau^2\sigma_i^2}\right)\frac{I_2}{\in}+\frac{g\alpha\kappa a^2}{\nu\beta(H^2d^4+\nu^2\tau^2\sigma_i^2)}\right.$$

$$\left\{d^2\nu\tau h I_3+(Hd^4+\nu^2\tau^2\sigma_i^2)\overline{E+h\in}\;p_1 I_4\right\}$$

$$\frac{g\alpha'\kappa'a^2}{\nu\beta'\left(H'^2d^4+\nu^2\tau^2\sigma_i^2\right.}$$

$$\left\{d^2\nu\tau h' I_5+(H'd^4+\nu^2\tau^2\sigma_i^2)\overline{E'+h'\in}\,p_1'\;I_6\right\}=0. \quad (33)$$

In the absence of solute gradient, eq. (33) yields $\sigma_i = 0$. This means that the principle of exchange of stabilities is valid for the couple-stress fluid, permeated with suspended particles, heated from below in porous medium. The stable solute gradient introduces oscillatory modes in the system.

6. Conclusions

The theory of couple-stress fluid has been formulated by Stokes [6] and allows for polar effects such as the presence of couple stresses and body couples. The synovial fluid has been modelled as a couple-stress fluid in human joints by Walicki and Walicka [7] whereas Rathod and Thippeswamy [8] have considered blood as couple-stress fluid and dust (suspended) particles as micro-organisms.

Sharma and Sharma [10] have studied a layer of couple-stress fluid heated from below in porous medium. The principle of exchange of stabilities is satisfied. For stationary convection, the couple-stress fluid postpones the onset of convection whereas the medium permeability hastens the onset of convection. A layer of electrically conducting couple stress fluid heated from below in porous medium in presence of magnetic field has been considered by Sharma and Thakur [11]. The magnetic field postpones the onset of convection and also introduces oscillatory modes in the system which were non-existent in its absence. A layer of couple-stress fluid heated from below and permeated with suspended particles has also been considered by Sharma, *et al* [12]. The principle of exchange of stabilities is also found to hold true in the presence of suspended particles, for non-porous medium.

A layer of couple-stress fluid, permeated with suspended particles, heated and soluted from below, in porous medium has been considered in the present paper. The principle of exchange of stabilities is found to hold good for the couple-stress fluid, permeated with suspended particles, heated from below (absence of solute gradient) in porous medium which is also true in the absence of suspended particles [11] and in the presence of suspended particles for non-porous medium [12]. The stable solute gradient, like magnetic field, postpones the onset of convection and introduces oscillatory modes in the system which were non-existent in its absence. Here the stabilizing effect of couple-stress and destabilizing effects of medium permeability and suspended particles are in agreement with the earlier studies [11,12].

References

[1] S Chandrasekhar *Hydrodynamic and Hydromagnetic Stability* (New York : Dover) (1981)

[2] K Chandra *Proc. Roy. Soc.* (London) **A164** 231 (1938)

[3] J W Scanlon and L A Segel *Phys. Fluids* **16** 1573 (1973)

[4] V I Palaniswamy and C M Purushotham *Phys. Fluids* **24** 123 (1981)

[5] D D Joseph *Stability of Fluid Motion* **Vol. II** (Springer-Verlag Berlin) (1976)

[6] V K Stokes *Phys. Fluids* **9** 1709 (1966)

[7] E Walicki and A Walicka *Appl. Mech. Engng.* **4** 363 (1999)

[8] V P Rathod and G Thippeswamy *Math. Edu.* **33** 40 (1999)

[9] G Veronis J. Marine *Res.* **23** 1 (1965)

[10] R C Sharma and S Sharma *Indian J. Phys.* **75B** 59 (2001)

[11] R C Sharma and K D Thakur *Czech. J. Phys.* **50** 753 (2000)

[12] R C Sharma, Sunil Divya and R S Chandel *Arch. Mech.* **54** 287 (2002)